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Knotted self-dual strings coupled with Kalb-Ramond field

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Abstract: It is shown that knotted topological strings appear at the symmetric phase of Higgs field in the context of generalized Yang-Mills Chern-Simons Higgs model. The equation of motion of the knotted strings coupled with the Kalb-Ramond field is obtained. Based on the equation of motion, we found that there is a topological constrain on the coupling term between Maxwell field and the covariant derivative of Kalb-Ramond field, the topological number of this coupling is the topological number of the abelian Chern-Simons action on the knotted topological strings.

Keywords: Topological Field Theories, Chern-Simons Theories, Topological Strings.

Contents

1. Introduction

Knots and strings play an important role in a variety of physical systems, such as the structure of elementary particles [1], the closed relativistic string in early universe cosmology [2], the knotted soliton in triplet superconductors [3], and so on. In a realistic (3+1)-dimensional model, there exist knotlike string structures appearing as stable finite solitons [4]. Topological strings in Chern-Simons theory has attracted increasing attentions [5] of physicists. Usually strings appear as solutions in the symmetry broken phase in four dimensional field theory with abelian symmetry. The most popular field-theoretical system used to model cosmic string generation is the abelian Higgs model [6]. In path integral formalism, Kim and Lee derived the dual formation of vortices in Chern-Simons Higgs system, extending this idea, Lee have developed the dual formation theory of cosmic strings [7]. The Higgs mechanism for Kalb-Ramond field provides a fundamental understanding to strings [8]. In four dimensions, the massless Kalb-Ramond field is dual to a spinless field ϕ which can be interpreted as an axion. On the other hand, a massive vector field in four dimensions is dual to a massive vector field. When the Kalb-Ramond field couples to the world sheet of string, the string condensation may realize the Higgs mechanism to the Kalb-Ramond gauge field [9]. Motivated by these works, we developed the topological theory of the formation of strings in Maxwell Higgs model in this paper, it is shown that the topological strings arise from the symmetric phase of Higgs field, while the non-topological strings arise from the asymmetric phase. For the topological knotted string, there is a topological constrain on the coupling term between the Maxwell field and the covariant derivative of the Kalb-Ramond field.

The paper is organized as follows: In section 2, By decomposing the Bogomol'nyi self-dual equations in 2+1 dimensional space time, we obtain the complete self-dual equation which includes a topological term which is not previously addressed in the literature. Then we generalize the self-dual condition into 3+1 dimensional space, and obtain the topological current of knotted strings. In section 3, the equation of motion of the knotted topological string coupled to Kalb-Ramond field is presented, based on it, we found the topological number of the coupling between the Maxwell field and the covariant derivative of Kalb-Ramond field. In the last section, we presented a brief summary and outlook.

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2. The knotted topological string in the symmetric phase of Higgs field

The novel features of topological strings and non-topological strings can be understood better by the abelian Chern-Simons Higgs model in 2+1-dimensions [10, 11]

$$L_{CSH} = \frac{1}{4}\kappa\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda} + ||D_\mu\phi||^2 + \frac{1}{\kappa^2}||\phi||^2(||\phi||^2 - v^2)^2\,, \tag{2.1}$$

where ϕ is the designated charged Higgs complex scalar field minimally coupled to an abelian gauge field, and the covariant derivative is defined as $D_\mu = \partial_\mu + ieA_\mu$. The first-order Bogomol'nyi self-dual equations [16] in this model is

$$D_\pm\phi = 0 \tag{2.2}$$

and

$$B = \pm\frac{2}{\kappa^2}||\phi||^2(||\phi||^2 - \nu^2)\,, \tag{2.3}$$

here the operator $D_\pm \equiv (D_1 \pm iD_2)$. The complex scalar field ϕ can be regarded as the complex representation of a two-dimensional vector field $\vec{\phi} = (\phi^1, \phi^2)$ over the base space time, i.e. $\phi = \phi^1 + i\phi^2$, where $\phi^a (a = 1, 2)$ are real functions. Substituting $\phi = \phi^1 + i\phi^2$ into the first case of the Bogomol'nyi self-dual equation $D_+\phi = 0$ and splitting the real part from the imaginary part, we obtain two equations

$$\begin{aligned} \partial_1\phi^1 - \partial_2\phi^2 &= eA_1\phi^2 + eA_2\phi^1 \\ \partial_1\phi^2 + \partial_2\phi^1 &= eA_2\phi^2 - eA_1\phi^1\,. \end{aligned} \tag{2.4}$$

Substituting the equations above into $\partial_1\phi^*\phi - \partial_1\phi\phi^*$, it yields

$$\partial_1\phi^*\phi - \partial_1\phi\phi^* = 2ieA_1||\phi||^2 + i(\partial_2\phi^*\phi + \partial_2\phi\phi^*)\,. \tag{2.5}$$

After we defined the unit vector as

$$n^a = \frac{\phi^a}{||\phi||}\,, \qquad ||\phi|| = (\phi\phi^*)^{\frac{1}{2}}\,, \tag{2.6}$$

using the formula

$$\frac{1}{2i}\frac{1}{\phi^*\phi}(\partial_i\phi^*\phi - \partial_i\phi\phi^*) = \epsilon^{ab}n^a\partial_i n^b\,, \tag{2.7}$$

we can rewrite eq. (2.5) as

$$eA_1 = \epsilon^{ab}n^a\partial_1 n^b - \frac{1}{2}\partial_2\ln(\phi\phi^*)\,. \tag{2.8}$$

When considering $\partial_2\phi^*\phi - \partial_2\phi\phi^*$ and eq. (2.4), following the same processes above, we have

$$eA_2 = \epsilon_{ab}n^a\partial_2 n^b + \frac{1}{2}\partial_1\ln(\phi\phi^*)\,. \tag{2.9}$$

Eq. (2.8) and eq. (2.9) can be unified into one equation

$$eA_\mu = \epsilon_{ab}n^a\partial_\mu n^b - \epsilon^{\mu\nu}\frac{1}{2}\partial_\nu\ln(\phi\phi^*)\,. \tag{2.10}$$

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Now let's turn to the second case of eq. (2.2), i.e., $D_-\phi = 0$, repeating the same discussions above, we arrive

$$eA_\mu = \epsilon_{ab} n^a \partial_\mu n^b + \epsilon^{\mu\nu} \frac{1}{2} \partial_\nu \ln(\phi\phi^*) \,. \tag{2.11}$$

So the gauge potential obtained from the Bogomol'nyi self-dual equation $D_\pm \phi = 0$ can be written as

$$eA_\mu = \epsilon_{ab} n^a \partial_\mu n^b \pm \epsilon^{\mu\nu} \frac{1}{2} \partial_\nu \ln(\phi\phi^*) \,. \tag{2.12}$$

In (2+1) dimensions, the magnetic field is given by $B = \epsilon^{ij}\partial_i A_j$, according to eq. (2.12), we have

$$B = \epsilon^{\mu\nu} \epsilon_{ab} \partial_\mu n^a \partial_\nu n^b \pm \nabla^2 \ln(||\phi||^2) \,.$$

For convenience, we denote $B_t = \epsilon^{\mu\nu}\epsilon_{ab}\partial_\mu n^a \partial_\nu n^b$ and $B_{nt} = \pm\nabla^2 \ln(||\phi||^2)$. According to Duan's ϕ-mapping topological current theory [15], using $\partial_\mu \frac{\phi^a}{\|\phi\|} = \frac{\partial_\mu \phi^a}{\|\phi\|} + \phi^a \partial_\mu \frac{1}{\|\phi\|}$ and the Green function relation in order parameter space: $\partial_a \partial_a (\ln \parallel \phi \parallel) = 2\pi\delta^2(\vec{\phi})$, $(\partial_a = \frac{\partial}{\partial \phi^a})$, we arrive

$$B_t = \epsilon^{\mu\nu} \epsilon_{ab} \partial_\mu n^a \partial_\nu n^b = 2\pi\delta^2(\vec{\phi}) J\left(\frac{\phi}{x}\right) \tag{2.13}$$

where $J(\phi/x) = \frac{1}{2}\epsilon^{\lambda\rho}\epsilon_{ab}\partial_\lambda \phi^a \partial_\rho \phi^b$ is the jacobian. This term describes the topological string excitation in Higgs field. So the Bogomol'nyi self-dual equation (2.3) can be rewritten as

$$B = 2\pi\delta^2(\vec{\phi}) J(\frac{\phi}{x}) \pm \nabla^2 \ln(\rho) = \pm \frac{2}{\kappa^2} \rho(\rho - v^2) \,, \tag{2.14}$$

where $\rho = ||\phi||^2$ is the scalar field density. When the Higgs field has no zero points, eq. (2.14) reduces to

$$B = \pm \nabla^2 \ln(\rho) \,. \tag{2.15}$$

This is the conventional self-dual equation [16]. It governs the non-topological string solutions. No exact solutions are known for this equation. In a special case, a vertex-like solution in three-dimensional space is given by A. Jaffe [17].

The self-dual potential $\frac{1}{\kappa^2}\rho(\rho - v^2)^2$ reveals the existence of two degenerate vacua: $\phi = 0$ and $||\phi|| = v$. Correspondingly there are domain wall solutions that interpolate between the two vacua [12]. In the Higgs vacuum, the Chern-Simons Higgs mechanism leads to a massive gauge field m_g and a massive real scalar field of mass m_s. The Chern-Simons Higgs model displays very different behavior depending on the relative magnitude of these two mass scales. Numerically, it has been shown that two strings repel if $m_s > m_g$, but attract $m_s < m_g$. when the masses are equal $m_s = m_g$, the forces between strings vanish and it is possible to find stable static multi-strings configurations.

While with the particular form of the self-dual potential $\frac{1}{\kappa^2}\rho(\rho - v^2)^2$, we have $m_s = 2v^2/\kappa = m_g$. So we can solve eq. (2.14) to obtain the multi-strings solutions. When system is in the symmetric phase $\phi = 0$, by solving equations $\{\phi^1(x^1, x^2, t) = 0, \phi^2(x^1, x^2, t) = 0\}$, we can derive the topological solutions. In this case, $B_{nt} = \pm\nabla^2 \ln(||\phi||^2)$ is singular, but this singularities can be cancelled by $B_t = 2\pi\delta^2(\vec{\phi})J(\frac{\phi}{x})$, so the complete magnetic field will remain nonsingular. When the system is in the asymmetric phase $\phi \neq 0$, the topological part of the magnetic field $B_t = 2\pi\delta^2(\vec{\phi})J(\frac{\phi}{x})$ vanishes, from the conventional self-dual equation, the non-topological solution is derived by neglecting the zeroes of ϕ [13]. But from eq. (2.14), we see, the solutions of the conventional self-dual equation has missed the information on the topological strings in the symmetric phase. So we mainly focus on the topological structure of the strings at the symmetric phase $\phi = 0$ in this paper.

The topological string can be viewed as a kind of topological defects. According to the Kibble mechanism for the formation of topological defects in the abelian Higgs model [14], when the vacuum bubbles collide, topological strings are formed during this phase transition. In a first-order phase transition, $\phi = 0$ remains a metastable fixed point below $T = T_c$, it transforms into a state with $\|\phi\| = v$(which represents a spherical configuration of vacuum,i.e., vacuum bubble) by bubble nucleation. As we know, instantons can tunnel through different vacua which are labelled by different symmetry. In the Chern-Simons Higgs model, the self-dual string plays a similar role as instantons. As mentioned above, there are domain wall solutions that interpolate between the two vacua [12], so the strings is separated into two classes by the domain wall. When the topological string tunnel through the domain wall and step into the asymmetric phase, it becomes the non-topological string. If the non-topological string step into the symmetric phase, it becomes the topological string, this process can be viewed as that the topological string is created through the Kibble mechanism. So the transition between the two classes of strings can be labelled by the phase transition from the symmetric phase to asymmetric phase and the inverse phase transition process. The Kibble mechanism provides us an insight into the generation of the topological string, there must be some other mechanisms to explain the annihilation of strings during the phase transition, this will be an open question which needs to be further studied.

The discussions above is based on the Chern-Simons Higgs model in 2+1 dimensions, we have chosen a cross section of the strings bundle for convenience, the motion of the strings is described by the points on the cross section of the world sheets of strings bundle. The topology of string itself has not been taken into account. There are many interesting properties originates from the topology of string itself. Especially for the closed knotted strings which differs from the open strings by linking numbers and self-linking numbers.

In order to study the topology of knotted strings itself, we introduce the generalized self-dual Chern-Simons model in (3+1)-dimensional space-time, which includes a Maxwell term for the gauge field, but this requires an additional neutral scalar field $\mathcal{N}$ [18]

$$L_M = \frac{-1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{2}\epsilon^{ijk}A_iF_{jk} + |D_\mu\phi|^2 + \frac{(\partial_\mu\mathcal{N})^2}{2e^2} - V(|\phi|,\mathcal{N})\,. \tag{2.16}$$

In the Yang-Mills Chern-Simons Higgs theory [11], the simplest self-dual gauge potential [19] which satisfies the gauge self-dual equation $F_{\mu\nu} = {}^*F_{\mu\nu}$ is of the form $A = f(r)ndn^\dagger$. When $f(r) = 1$, this gauge potential can be obtained from the gauge condition $D_\mu\phi = 0$, $(\mu = 0, 1, 2, 3)$ [20]. This gauge condition could be viewed as the generalization of the self-dual equation $D_\pm\phi \equiv (D_1 \pm iD_2)\phi = 0$ into 3+1 dimensional space time, in which the dual operator of $D_\pm \equiv (D_\mu \pm iD_\nu)$ would be defined as ${}^*D_\pm \equiv \epsilon^{\mu\nu\lambda\rho}(D_\lambda \mp iD_\rho)$, here $\lambda\rho$ does not obey Einstein summation rule. From the two operators $D_\pm$, we have $D_\mu\phi = \frac{1}{2}(D_+ + D_-)\phi = 0$; $D_\nu\phi = \frac{1}{2i}(D_+ - D_-)\phi = 0$, it is in consistent with $D_\mu\phi = 0$.

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Since $B^i = \epsilon^{ijk}F_{jk}$, we can treat $F^{\mu\nu}$ as the generalization of the magnetic field B. Applying the same decomposition method with $D_\pm\phi \equiv (D_\mu \pm iD_\nu)\phi = 0$ $(\mu,\nu = 0,1,2,3)$, we can also obtain the complete self-dual equation $F = 2\pi\delta^2(\vec{\phi})J(\frac{\phi}{x}) \pm \nabla^2 ln(\rho)$. But if we choose the gauge condition $D_\mu\phi = 0$, it is easy to verify that $A_\mu = \epsilon_{ab}n^a\partial_\mu n^b$, substitute it into the U(1) gauge field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, it yields $F = 2\pi\delta^2(\vec{\phi})J(\frac{\phi}{x})$. Now we see the non-topological part $\pm\nabla^2 ln(\rho)$ is eliminated.

Therefore the filter gauge condition $D_\mu\phi = 0$ in 3+1 dimensional space could be viewed as the generalized self-dual condition $D_\pm\phi = 0$ in 2+1 dimensional space time, and it has filtered out the non-topological part. In this paper, we mainly focus on the topological part $F = 2\pi\delta^2(\vec{\phi})J(\frac{\phi}{x})$, since it is not previously addressed in the literature.

In the dual formulation, a global string appears as the source of an antisymmetric tensor field, which represents the Goldstone boson [7]. In the Yang-Mills Chern-Simons Higgs model, the topological strings arise from the gauge field tensor

$$F_{\mu\nu} = 2\epsilon_{ab}\partial_\mu n^a\partial_\nu n^b \tag{2.17}$$

which describes the non-trivial distribution of $\vec{n}$ at large distances in space [21]. We define the topological current of strings as

$$j^{\mu\nu} = \frac{1}{8\pi}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}. \tag{2.18}$$

It can be proved to be

$$j^{\mu\nu} = \delta^2(\vec{\phi})J^{\mu\nu}\left(\frac{\phi}{x}\right), \tag{2.19}$$

where $J^{\mu\nu}(\phi/x) = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\epsilon_{ab}\partial_\lambda\phi^a\partial_\rho\phi^b$ is the jacobian vector. The spatial component of the current $j^{\mu\nu}$ is

$$j^{0i} = j^i = \frac{1}{8\pi}\epsilon^{ijk}F_{jk}. \tag{2.20}$$

This expression provides an important conclusion: $j^i = 0$, iff $\vec{\phi} \neq 0$; $j^i \neq 0$, iff $\vec{\phi} = 0$. So the topological string are situated at the symmetric phase of the Higgs field $\phi = 0$. To determine the non-zero solutions of the topological current j^i, we need to solve the equations $\phi^1 = 0$, $\phi^2 = 0$. The implicit function theory shows [22] that under the regular condition $D^i(\phi/x) \neq 0$, the general solutions of

$$\phi^1(t,x^1,x^2,x^3) = 0, \phi^2(t,x^1,x^2,x^3) = 0 \tag{2.21}$$

can be expressed as

$$x^1 = x^1_k(s,t), x^2 = x^2_k(s,t), x^3 = x^3_k(s,t) \tag{2.22}$$

which represent the world sheets of N moving isolated singular strings L_k with string parameter s $(k = 1,\ldots,N)$. These singular string solutions can be viewed as the cosmic strings, they also arise as the Mermin-Ho vortex lines in BEC [23]. In the spatial 3-dimensions, it can be proved that [24, 15]

$$j^i = \delta^2(\vec{\phi})J^i(\frac{\phi}{x}) = \sum_{k=1}^{N} W_k \int_{L_k} \frac{dx^i}{ds}\delta^3(\vec{x} - \vec{x}_k(s))ds\,, \tag{2.23}$$

where $W_k = \beta_k \eta_k$ is the winding number of $\vec{\phi}$ around L_k, with the positive integer β_k being the Hopf index and $\eta_k = \pm 1$ the Brouwer degree of ϕ-mapping. Then from eq. (2.23) one obtains the topological charge of string L_k: $Q_k = \int_{\Sigma_k} j^i d\sigma_i = W_k$.

The strings in the configuration space can be closed, open or half-open. If there exist magnetic monopoles in space time, the open strings is attached by monopoles and anti-monopoles at both ends, the half-open string is attached by a monopoles at one side. We shall study a family of closed knotted strings γ_k $(k = 1, \ldots, N)$. Using the $\phi-$mapping topological current theory, it can be proved that the the Chern-Simons action $S_{CS} = \frac{1}{8\pi} \int_M \epsilon^{ijk} A_i F_{jk}$ is a topological invariant on these knotted singular strings, and the topological number is given by [25]

$$Q = 2\pi \left[\sum_{k=1}^{N} W_k^2 SL(\gamma_k) + \sum_{k,l=1\ (k \neq l)}^{N} W_k W_l Lk(\gamma_k, \gamma_l) \right], \tag{2.24}$$

where $SL(\gamma_k)$ is the self-linking number of γ_k following the White formula $SL(\gamma_k) = Wr(\gamma_k) + Tw(\gamma_k)$. $Lk(\gamma_k, \gamma_l)$ is the Gauss linking number between knotted strings γ_k and γ_l $(k \neq l)$. This topological number Q is also equal to Faddeev-Niemi number [1]. It will be shown in the following that this topological number presents a constrain on the Kalb-Ramond field in the equation of motion.

3. Knotted string coupled with Kalb-Ramond field

When a point-like fermion field interact with Maxwell and Kalb-Ramond fields, the vacuum polarization to fermions will induce the coupling between Maxwell and Kalb-Ramond fields [26], in this case, the simplest lagrangian which is invariant under U(1) gauge transformation is

$$\mathcal{L}_{\text{fermi}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{12} \mathcal{H}_{\mu\nu\lambda} \mathcal{H}^{\mu\nu\lambda} + \psi^\dagger \left(D - \frac{g}{12m} \sigma_{\mu\nu\lambda} \mathcal{H}^{\mu\nu\lambda} - m \right) \psi\,. \tag{3.1}$$

As seen in the complete self-dual equation (2.14), the topological vortex in 2+1-dimensional space time is generated from the symmetric phase of the bosonic Higgs field. So according to eq. (2.18), the knotted topological string also arise from the symmetric phase of the Higgs field in four dimensional space. In analogy with lagrangian of K-R field coupled with Maxwell field in the context of fermion field, we generalize Nielsen's lagrangian of strings [27] and arrive the bosonic lagrangian

$$L_{K-R} = \sqrt{g} \Big[-\mu \sqrt{(j, j)} + \xi (j, \mathcal{B}) - \frac{1}{2} (\mathcal{H}, \mathcal{H}) \Big], \tag{3.2}$$

in which the topological current $j = {}^*F$ is given by the dual field of the 2-form Maxwell field $F = \frac{1}{2\pi} \epsilon_{ab} dn^a \wedge dn^b$, i.e.,

$$j = {}^*F = \frac{1}{2} \epsilon_{ab} \epsilon^{\mu\nu}{}_{\lambda\rho} \partial_\mu n^a \partial_\nu n^b dx^\lambda \wedge dx^\rho \tag{3.3}$$

with the Hodge star operator $*$. The second term of right hand side of the lagrangian (3.2) is the coupling action between the Maxwell field and 2-form Kalb-Ramond field $\mathcal{B}$. In the local coordinates the Kalb-Ramond field $\mathcal{B}$ is expressed as $\mathcal{B} = \frac{1}{2}\mathcal{B}_{\mu\nu}dx^\mu \wedge dx^\nu$. $\mathcal{H}$ is 3-form field $\mathcal{H} = d\mathcal{B} = \frac{1}{3!}\mathcal{B}_{\mu\nu\lambda}dx^\mu \wedge dx^\nu \wedge dx^\lambda$ with its components as $\mathcal{H}_{\mu\nu\lambda} = \partial_\mu\mathcal{B}_{\nu\lambda} + \partial_\nu\mathcal{B}_{\lambda\mu} + \partial_\lambda\mathcal{B}_{\mu\nu} = 3\partial_{[\mu}\mathcal{B}_{\nu\lambda]} = 3\nabla_{[\mu}\mathcal{B}_{\nu\lambda]}$.

From eq. (2.18) and eq. (2.19), it is easy to see that the current j of topological strings can be written as $j = \delta(\phi)J$, in which $J = \frac{1}{2}J_{\mu\nu}(\frac{\phi}{x})dx^\mu \wedge dx^\nu$ with the coefficient $J_{\lambda\rho}$ denoted as $J_{\mu\nu} = \frac{1}{2}\epsilon_{ab}\epsilon_{\mu\nu}{}^{\lambda\rho}\partial_\lambda\phi^a\partial_\rho\phi^b$. So the interaction term between the knotted topological strings of the lagrangian is expressed as

$$L_{\text{string}} = -\mu\sqrt{(j,j)} = -\mu\delta(\phi)\sqrt{(J,J)}\,. \tag{3.4}$$

In the De Rham current theory [28], the inner product of the two currents here is defined as $(J,J) = \frac{1}{2}J_{\lambda\rho}J^{\lambda\rho}$.

For convenience, we split the lagrangian (3.2) into two parts $L_{K-R} = L_1 + L_2$ with $L_1 = -\sqrt{g}[\mu\sqrt{(j,j)} + \xi(j,\mathcal{B})]$ and $L_2 = -\sqrt{g}[\frac{1}{2}(\mathcal{H},\mathcal{H})]$. From eq. (3.4), one sees that the topological strings arise from the symmetric phase of the Higgs field, their location is given by the solutions of the equation $\phi^a(x^\mu) = 0, (\mu = 0,1,2,3; a = 1,2)$. As the system evolves from the symmetric phase into the asymmetric phase, the L_1 lagrangian term disappeared for the non-topological strings. So this lagrangian L_1 reflects the phase transition of the system, we can study the two phases from the equation of motion which arise from this term.

In quantum field theory, the equation of motion is obtained from the variational derivative of the action. Firstly, we need to calculate the energy momentum tensor which is given by $T^{\mu\nu} = -2(\partial(\sqrt{-g}L))/(\sqrt{g}\partial g_{\mu\nu})$, it can be correspondingly split into two parts $T^{\mu\nu} = T^{\mu\nu}_{(1)} + T^{\mu\nu}_{(2)}$, where $T^{\mu\nu}_{(1)}$ is

$$T^{\mu\nu}_{(1)} = -\mu\delta(\phi)|J_0|g^{IJ}\frac{\partial x^\mu}{\partial u^I}\frac{\partial x^\nu}{\partial u^J} - 2\xi\delta(\phi)J_0\frac{\partial x^\mu}{\partial u^I}\frac{\partial x^\nu}{\partial u^I} \tag{3.5}$$

in which J_0 is the jacobian and g^{IJ} is the metric on the world sheets of strings with two string parameter $u^1 = s$ and $u^2 = t$, which correspond respectively to the string parameter and the time parameter. From eq. (3.5) and the conserved energy-momentum tensor

$$\nabla_\mu T^{\mu\nu}_{(1)} = 0\,, \qquad \nabla_\mu T^{\mu\nu}_{(2)} = 0\,, \tag{3.6}$$

we arrive the dynamic equation of motion

$$\frac{1}{\sqrt{-g_N}}\partial_I(\sqrt{-g_N}g_N{}^{IJ}\partial_J x^\lambda) + \Gamma^\lambda_{\mu\nu}g_N{}^{IJ}\partial_I x^\mu\partial_J x^\nu = \eta\frac{\xi}{\mu}\frac{\partial x^\mu}{\partial u^l}\frac{\partial x^\nu}{\partial u^l}\mathcal{H}_\mu{}^\lambda{}_\nu\,. \tag{3.7}$$

This equation plays a similar role as the geodesic equation in curved space time. Taking the variational derivatives of the action with respect to $\mathcal{B}$, it yields $\nabla_\lambda\mathcal{H}^{\lambda\mu\nu} = \xi j^{\mu\nu}$, i.e.,

$$\frac{1}{\sqrt{-g}}\partial_\lambda(\sqrt{-g}\mathcal{H}^{\lambda\mu\nu}) = \xi j^{\mu\nu}\,, \tag{3.8}$$

it can be expressed in a more elegant form: $\delta\mathcal{H} = \xi j$, this is the dynamic equation of strings coupled with the Kalb-Ramond field. In fact, the current $j^{\mu\nu}$ appears as the radiation sources of the the Kalb-Ramond field. This can be seen more clearly from the De Rahm current form of the equation of motion. When choosing the Lorentz gauge $\nabla_\mu \mathcal{B}^{\mu\nu} = 0$, i.e., $\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\mathcal{B}^{\mu\nu}) = 0$, we can rewrite equation $\delta d\mathcal{B} = \xi j$ as $(\delta d + d\delta)\mathcal{B} = \xi j$, here we have used $\delta\mathcal{B} = 0$ which is the equivalent expression of the Lorentz gauge in the language of De Rahm current. Recall the Laplace operator $\triangle = \delta d + d\delta$, we can reduce eq. (3.8) into a more elegant form in terms of the language of De Rahm current: $\triangle\mathcal{B} = \xi j$. This equation is of the same form as the generalized Maxwell equation in the electrodynamics, one can easily see that the global cosmic string does acts as the sources radiating the 3-form field $\mathcal{H}$. It means when the string is coupled with background field, the global cosmic string will radiate Goldstone Bosons.

Further more, the topological number of the knotted strings plays an important role in the equation of motion. As mentioned above, the Chern-Simons action is a topological invariant on these knotted topological strings arise from the symmetric phase of Higgs field. In fact, the Chern-Simons action $S_{CS} = \frac{1}{8\pi}\int_M \epsilon^{ijk} A_i F_{jk}$ can be expressed by the spatial component of the current $j^{\mu\nu}$ of strings as $S_{CS} = \int_M A_i j^i$, where $j^i = j^{0i} = \frac{1}{8\pi}\epsilon^{ijk}F_{jk}$. This current j^i is governed by the equation of motion (3.8). Substituting the spatial equation of eq. (3.8) into the Chern-Simons action, we arrive

$$\int_M \frac{1}{\xi\sqrt{-g}} A_i \partial_\lambda(\sqrt{-g}\mathcal{H}^{\lambda 0 i}) d^3x = Q\,, \tag{3.9}$$

where Q is given by eq. (2.24), it is consist of the self-linking numbers of the knotted string itself and linking numbers between knotted strings, they both are topological invariant characteristic numbers of knotted configurations in geometry. This topological number is conserved during the generation, annihilation and bifurcation of knotted strings. So there is a topological constrain on the Kalb-Ramond field during the evolution of knotted strings.

In fact, the topological number Q also connote the information of the asymmetric phase of the Higgs field. In quantum field theory, the global symmetry spontaneous breaking gives birth to the Goldstone Bosons [29] which was imbedded in the scalar field ϕ. When a knotted cosmic string peers through the area Σ in space time on which the Higgs field $\phi \neq 0$, it satisfies the dual equation $\partial_\mu\phi = \frac{1}{6}\epsilon_{\mu\nu\lambda\rho}\mathcal{H}^{\nu\lambda\sigma}$. The integral can be carried out along with the boundary of $\partial\Sigma = C$ with C as a 1-chain, it yields

$$\oint_C \partial_\mu\phi dx^\mu = \oint_C *\mathcal{H} = \int_\Sigma d*\mathcal{H}\,, \tag{3.10}$$

From eq. (2.23) and eq. (3.8), we have $d*\mathcal{H} = \xi d\omega$, then

$$\oint_C \partial_\mu\phi dx^\mu = \xi\int_\Sigma d\omega = W\xi\,, \tag{3.11}$$

where W is the winding number around the string and ξ is the unit quantization. This is just the quantization condition of string, it is similar to the quantized vortices in superfluid. Noticing here the integral is carried out in the asymmetric phase of the Higgs field, so the quantized number of the symmetric phase carries the information of asymmetric phase.

4. Conclusion

When decomposing the Bogomol'nyi self-dual equation 2+1 dimensional space time, we arrive the complete Bogomol'nyi self-dual equation, it includes a topological term and a non-topological term, in which the topological terms is not previously addressed in the literature. Based on the complete self-dual equation, it is shown that the topological strings arise from the symmetric phase of Higgs field, non-topological string arise from the asymmetric phase, so the phase transition between the symmetric phase and the asymmetric phase can be distinguished by the magnetic field. In analogy with the case in 2+1 dimensional space time, we obtained the generalized filter self-dual equation $D_\mu \phi = 0$ in 3+1 dimensional space time, and find a topological current of knotted self-dual string. It is shown that the topological string appear at the symmetric phase of the Higgs field $\phi = 0$. This provides us with a deep insight into the Higgs mechanism in Kalb-Ramond field. More over, we shown that a family of knotted strings are attached by a topological invariant: the abelian Chern-Simons action. When taking into account of the coupling between the knotted string and the Kalb-Ramond field, we got the dynamic equation of motion of these knotted strings, and found that these strings act as the sources radiating Kalb-Ramond field. Further more, the coupling between the Maxwell field and the covariant derivative of the Kalb-Ramond field is governed by a topological number: the Faddeev-Niem number, it is also the topological number of the abelian Chern-Simons action.

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